Waveguide Step Discontinuities Revisited by the Generalized Network Formulation

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Abstract— **We consider the classical problem of waveguide step discontinuities from the perspective of the generalized network formulation. The latter has recently been introduced for systematically dealing, in an efficient and rigorous manner, with electromagnetic field representations and computations in complex structures. The approach is based on the topological partitioning of the complex structure into several subdomains joined together by interfaces. The suggested framework accommodates the use of different analytical/numerical methods (hybridization), the choice of alternative Green's functions and the selection of appropriate field quantities at the boundary between different regions.**
By using the generalized network formulation in the step discontinuity example we note that it is possible to select alternative Green's functions with improved convergence properties with respect to those commonly used. In addition, a new canonical representation of the step discontinuity is derived and better insight is obtained on the relationship between integral equation formulations and mode-matching techniques for the analysis of step discontinuities.

I. Introduction

Waveguide step discontinuities are the basic building blocks for the numerical analysis of complex waveguide components. The current trend of designing such components with very high performance, possibly also taking into account manufacturability issues, demands the use of computer-intensive optimization programs. When using such codes for design purposes, the full-wave analysis of basic discontinuities is performed several thousand of times, and it is therefore necessary to develop very efficient routines for this task. Not surprisingly, step discontinuity problems have received considerable attention in the past (see e.g. [1, chap. 5], [2]). Due to the separability of the wave equation in the waveguide subsections [3], essentially two types of approaches have been developed: one based on mode-matching at the step discontinuity and the other based on an integral equation formulation.

Recently, a general architecture has been proposed [4], [5], [6], [7], [8] for the systematic electromagnetic field computation in complex structures. This approach is based on the topological partitioning of the complex structure into several subdomains joined together by interfaces. The basic concept is to describe in a separate manner the topological relationship (how the subdomains are connected to each other) and the physical relationships. Apart from accommodating the use of different analytical/numerical methods (hybridization), this approach also deals in a systematic manner with a given problem, thus providing new insights also for rather well-known problems. In particular, in this study, we derive by this approach both the mode-matching and the integral equation formulations; in both cases we are also able to introduce some novelties which yield improvements in the numerical efficiency and in the physical understanding.

Integral equation techniques have permitted introduction of basis functions which include the edge condition [9], [10] and of the admittance matrix formulation [11], [12]. In these cases, however, the choice of the pertinent Green's function in the waveguide subregions was conventional, corresponding to an eigenfunction expansion in the transverse direction and waves propagating (and reflected) in the longitudinal direction. Accordingly, slowly convergent sums were obtained for steps with significantly different aspect ratios. In this paper we present an alternative Green's function expression which overcomes this problem and allows us to use rapidly convergent sums also for fairly high aspect ratios. The theory and numerical results for this case are briefly summarized in §II.

Mode-matching considers two different field expansions at the step itself, i.e. in a region of zero volume; in this case mode coupling arises at the step discontinuity and one seeks a description of the step discontinuities. It has been found that, although several alternatives are available, a description of the type employed in [13], [14] is necessary in order to obtain meaningful and accurate results. In this approach the independent field quantities are the electric field in the waveguide with the smaller cross-section and the magnetic field in the waveguide with the larger cross-section. This fact has been recently explained in [15] but no multimodal equivalent circuits have been provided so far. In fact, the only rigorous full-wave multi-mode frequency-independent equivalent circuit published for the step discontinuity [13], [14] makes use of controlled sources. Here, in section § III, by considering the step discontinuity as a connection network, we introduce a new canonical network based solely on transformers.

Finally, in section § IV, we discuss the relationship between mode-matching techniques and the integral equation approach. It is shown that the integral equation technique is a more general approach. In particular, when selecting a particular type of Green's function and the same number of modal expansion terms, the integral equation technique and

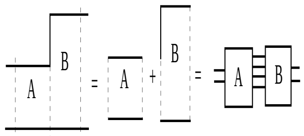

Fig. 1. Step discontinuity segmentation: a subdivision in two regions of space (typically used in admittance formulations). The first and last reference planes (dotted vertical lines) indicate the location for the "accessible" field variables.

the mode-matching approach provide the same results.

II. Use of alternative Green's functions in integral equation formulations

We start with subdividing our geometry, i.e. the waveguide step discontinuity, into a number of subdomains (see Fig. 1) which may be of different types, and which are joined together across interfaces. It is apparent that several different topological alternatives are available: we will not investigate these alternatives in this paper. In this section we consider a subdivision into two regions of space, see Fig. 1, which leads to the integral equation (admittance) formulation.

A. Evaluation of the admittance elements

Let us consider the waveguide step discontinuity illustrated in Fig. 2. Essentially, by applying the equivalence theorem, we place on the discontinuity section a p.e.c. with equivalent magnetic currents; we then evaluate the magnetic field generated on both sides and impose the continuity of its tangential components. Typically, a Galerkin discretization procedure is adopted and the modes of the smaller waveguide are chosen as the basis function set. Consequently, most of the numerical effort is devoted to computing the elements of the admittance matrix y^2_{np} in region B of Fig. 1, representing the magnetic field tested by the n-th weighting function as generated by the p-th electric field basis function. Usually, for their evaluation, an eigenfunction expansion in the y direction is chosen, providing the following representation of the Green's function

$$G^y = \sum_{m=0}^{\infty} g_m(y) g_m(y') \frac{\cos(k_{zm} z_<) \cos(k_{zm}(c - z_>))}{k_{zm} \sin(k_{zm} z_<)}$$
$$g_m(y) = \sqrt{\frac{\epsilon_m}{b_2}} \cos(\frac{m\pi}{b_2} y) \qquad (1)$$

Where $k^2_{zm} = k^2_0 - (\frac{\pi}{a})^2 - (\frac{m\pi}{l})^2$. This choice, however, generates the problem of "relative convergence" [16], [17], i.e. the number of terms to be used in the Green's function expansion depends on the ratio b_2/b_1. The larger this aspect ratio, the larger is the number of terms to be considered for the Green's function representation.

The problem of relative convergence can be overcome by considering an alternative Green's function representation which emphasizes wave propagation (and reflection) in the y direction and modal expansion in the z direction. In this case the Green's function takes the form

$$G^z = \sum_{m=0}^{\infty} f_m(z) f_m(z') \frac{\cos(k_{ym} y_<) \cos(k_{ym}(c - y_>))}{k_{ym} \sin(k_{ym} y_<)}$$
$$f_m(z) = \sqrt{\frac{\epsilon_m}{l}} \cos(\frac{m\pi}{l} z) \qquad (2)$$

Where $k^2_{ym} = k^2_0 - (\frac{\pi}{a})^2 - (\frac{m\pi}{b_2})^2$. By using these two different Green's function representations in the evaluation of

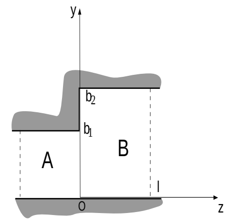

Fig. 2. Geometry of a E-plane step discontinuity between two rectangular waveguides of width a in the x direction. The computational domain lies between the two dotted vertical lines.

the admittance terms, we get the following expressions with different convergent properties:

$$Y^y_{n,p} = j\omega\epsilon \sum_{m=0}^{\infty} \frac{\sqrt{\epsilon_n \epsilon_p}}{b_1} \frac{\epsilon_m}{b_2} \frac{\cos(k_{zm} l)}{k_{zm} \sin(k_{zm} l)} \frac{(-1)^{n+p} k^2_m}{(k^2_p - k^2_m)(k^2_n - k^2_m)} \sin^2(k_m b_1) \qquad (3)$$

Where $k_j = (\frac{j\pi}{b_1})$ $j = p, n$ and $k_m = (\frac{m\pi}{b_2})$.

$$Y^z_{n,p} = j\omega\epsilon \sum_{m=0}^{\infty} \frac{\sqrt{\epsilon_n \epsilon_p}}{b_1} \frac{\epsilon_m}{l} \frac{k_{ym}(-1)^{n+p}}{(k^2_n - k^2_{ym})(k^2_p - k^2_{ym})} \frac{\sin(k_{ym} b_1) \sin(k_{ym}(b_2 - b_1))}{\sin(k_{ym} b_2)} + j\omega\epsilon \frac{\sqrt{\epsilon_n \epsilon_p}}{b_1} \delta_{n,p} \frac{b_1}{\epsilon_n} \frac{\cot\left(l\sqrt{k^2_0 - k^2_p}\right)}{\sqrt{k^2_0 - k^2_p}} \qquad (4)$$

A.1 Static part extraction

In both expression (3) and (4) we need to evaluate the sums in order to compute the admittance terms. It is well known that, for wide-band evaluation, a different arrangement of these sums is often convenient. In fact, denoting by S one of the above sums evaluated at a certain given frequency, we can write the generic admittance term, Y, evaluated at a different frequency, as given by $Y = S + (D - S)$; here D represents the sum evaluated a the frequency of interest. It is noted that the elements appearing in the sum $(D - S)$ converge very rapidly. This is a well known technique, generally applied with S representing a static term and D the dynamic contribution. It is noted that this useful device can be applied in the evaluation of both expressions (3) and (4).

B. A numerical example

As an example, in Fig. 3 we show the convergence behavior of one element of the admittance matrix with respect to m, i.e. with respect to the number of terms used to represent the Green's functions in (1) and (2). From the figure it is apparent that a significant advantage is obtained when considering the proposed alternative representation instead of the usual Green's function expression.

Similarly, in Fig. 4 we illustrate the convergence behavior for the same case, but including the static part extraction. Clearly, this accelerates the rate of convergence. Thus, using this device and the appropriate alternative Green's function selection, convergence is achieved with just a few terms.

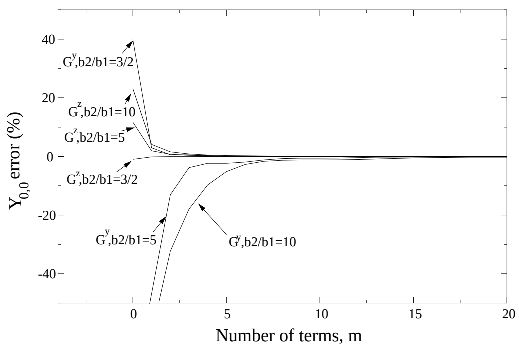

Fig. 3. Convergence behavior of the element $n = 0, p = 0$ of the admittance admittance matrix. The waveguide width is $a = 19mm$ and m is the number of terms considered in the sum in eq. (3) and (4). It is apparent that the usual Green's function, G^Y, converges relatively slowly and with a strong dependence on the geometrical ratio b_2/b_1. On the contrary, the alternative Green's function G^Z, which emphasize propagation and reflection in the y direction and modal expansion in the z direction, converges rapidly.

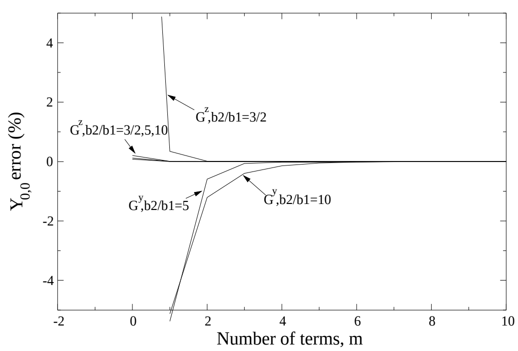

Fig. 4. As in Fig. 3 but with static-part extraction.

III. The step discontinuity as a connection network (Mode-matching)

Let us consider a partioning of the step discontinuity into two regions of space plus a connection network, see Fig. 5; this is a subdivision which leads to a mode-matching formulation, based on the field representation problem arising at the step discontinuity [18]. In order to investigate this problem it is convenient to refer to the bifurcation shown in Fig. 6 where three different subdomains are joined together. In particular, there is an interface which connects subdomain 1 to subdomain 3, and an interface connecting subdomain 2 to subdomain 3. In the following, for brevity, we assume that the electric (magnetic) fields at the interfaces are expanded

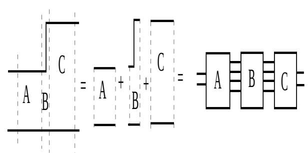

Fig. 5. Step discontinuity segmentation: a subdivision into three different regions (the central one being of zero volume, i.e. a connection network) as generally used in mode-matching formulations.

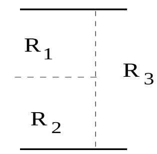

Fig. 6. The bifurcation problem: three regions of space connected at an interface.

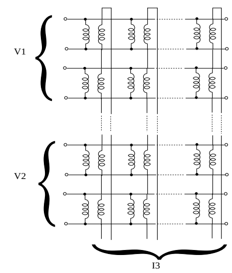

Fig. 7. A canonical network for the bifurcation: with $\mathbf{V_1}, \mathbf{V_2}$ and $\mathbf{I_3}$ chosen as independent field quantities.

in terms of suitable basis functions and we denote by $\mathbf{V_i}$ ($\mathbf{I_i}$) the vector containing the electric (magnetic) field expansion coefficients relative to region i.

It has been shown elsewhere [19] that the connection network for this interface can be obtained by taking $\mathbf{V_1}, \mathbf{V_2}$ and $\mathbf{I_3}$ as independent variables leading to the canonical network representation in Fig. 7. The other choice of independent variables is $\mathbf{I_1}, \mathbf{I_2}$ and $\mathbf{V_3}$ which leads to a similar canonical network. Both representations are equally valid in order to describe the connection network relative to a bifurcation.

However, in the case of the step discontinuity, region 1 is filled by a p.e.c., represented by a short-circuit. Thus we need to impose the condition $\mathbf{V_1} = \mathbf{0}$. The equivalent network is now the one in Fig. 7 with the ports pertaining to region 1 short-circuited.

It is useful to note that the above canonical network is frequency independent, satisfies the Tellegen theorem and admits a scattering representation with the following properties: symmetry, $\mathbf{S^T} = \mathbf{S}$, orthogonality, $\mathbf{S^T S} = \mathbf{I}$ and unitary, i.e. $\mathbf{S}\mathbf{S}^\dagger = \mathbf{I}$, where the $\dagger$ denotes the hermitian conjugate matrix, T denotes the transposed matrix and $\mathbf{I}$ is the identity matrix.

Also note that the above discussion is valid in general, for any choice of basis functions in regions 2 and 3. In practice, the most common choice of basis functions is the use of the modal eigenfunctions at both sides of the discontinuity; moreover it is common to place the reference planes at a certain distance from the discontinuity itself. We have therefore a certain number of modes which propagate from the discontinuity itself to the reference planes and are represented by transmission lines; by contrast the modes well below cut-off provide a localized contribution only at the discontinuity itself and can be represented by lumped, frequency-dependent reactances. It is also noted that the model proposed in this contribution, similarly to the model proposed in [13], [14] can be easily implemented in standard circuit simulators.

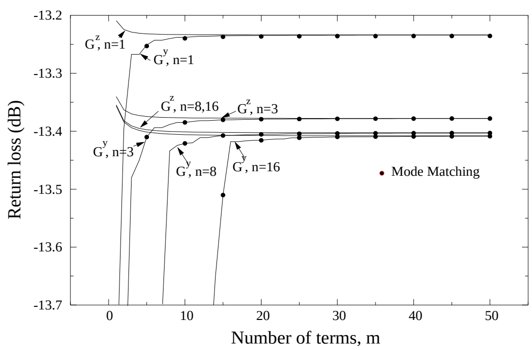

Fig. 8. Convergence for the scattering parameter when considering a n modes in the smaller waveguide and m terms for the Green's function, or m modes in the larger waveguide. In the integral equation approach we can select two different types of Green's function: the standard one (here denoted by G^y) and the one here introduced G^z. It is noted that the mode matching approach provides the same results of the integral equation approach when G^y is used as Green function.

A. Some numerical Results

In Fig. 8 we have plotted the convergence of the scattering parameter for a given step discontinuity when considering n modes in the smaller waveguide and m modes in the larger waveguide. It has been observed in [15] that m should be larger than n; it is also known that their ratio should be approximately equal to b_2/b_1. This is a useful criteria for fairly simple geometries; however, for other types of step discontinuities (e.g. the discontinuity between a small rectangular waveguide and a fairly large elliptical waveguide, as in [20]) it is difficult to select "a priori" the value of m.

IV. Relation between the integral equation approach and mode-matching

From the previous two sections it is apparent that the integral equation approach is more general than the mode-matching formulation. While in the integral equation approach we can choose between alternative Green's functions, the latter possibility is not present in mode-matching.

If we select the Green's function in eq. (1) and the same number of modal expansion terms m, then the integral equation technique and the mode-matching approach provide the same results (see Fig. 8. However, it has been noted that, for large aspect ratios, this choice is not the most convenient one. Moreover, while when considering the integral equation (Green's functions) it is straightforward to separate the static and dynamic contributions, as shown in § II-A.1, this does not seem to be the case for the mode-matching.

On the other hand, the mode-matching formulation provides a canonical, frequency-independent, equivalent circuit suitable for implementation in circuit simulators.

V. Conclusions

By applying some of the concepts developed for a generalized network formulation of EM field computation in complex structures we have found some new results also for fairly well-known problems such as the step discontinuity.

In particular, we have found that the systematic use of alternative Green's functions can significantly improve the convergence properties of modal sums. We have also found a canonical network representation for the step discontinuity and discussed the relationship between the integral equation approach and the mode-matching technique.

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